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CONSTRUCTION OF OPTIMAL POSITION STRATEGIES IN A DIFFERENTIAL PURSUIT–EVASION GAME WITH ONE PURSUER AND TWO EVADERS†

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The game-theoretic pursuit–evasion problem of one pursuer and two evaders is considered. It is assumed that one of the evaders must leave the game (disappear) at some time; however, neither this time nor the leaving evader is known in advance. The dynamics of all the objects can be described by the equations of the well-known Isaacs problem of the "game of two cars" [1] subject to conditions of restricted manoeuvrability of the objects. The minimum time difference between the pursuer and the evader remaining in the game is the payoff of the game. Under certain assumptions, relating the parameters of the objects and their initial positions, the optimal position strategies for the pursuer and two evaders are constructed. The formal description of the problem follows that considered in [2]. The approach proposed in [3] is developed. Similar problems were considered in [10–16].
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1. EQUATIONS OF MOTION AND PAYOFF FUNCTIONAL

Suppose that objects P (the pursuer) and E and E_1 (the evaders) move with constant velocity in the XOY plane (Fig. 1). The equations of motion of the objects and the constraints on their control interactions resemble those in the well-known problem of "the game of two cars" [1] and have the form

$$\dot{x}_i = V_i \sin\theta_i, \quad \dot{y}_i = V_i \cos\theta_i, \quad \dot{\theta}_i = (V_i / R_i)\varphi_i, \quad |\varphi_i| \leqslant 1 \tag{1.1}$$

Here V_i is the constant velocity, R_i is the minimum radius of curvature of the trajectory, θ_i is the angle between the OY axis and the vector V_i, φ_i is the scalar control function, C_i is the centre of curvature of the trajectory, and $i = 1$ corresponds to P, $i = 2$ to E and $i = 3$ to E_1.

It is assumed that

$$V_2 = V_3, \quad R_2 = R_3 \tag{1.2}$$

We will assume that the following restricted manoeuvrability condition is satisfied for system (1.1) ($\delta\theta_i$ being sufficiently small)

$$\theta_i = \theta_i^0 + \delta\theta_i \tag{1.3}$$

where θ_i^0 is the value of θ_i at the initial instant of time $t = t_0$.

Under this assumption, P, E and E_1 will move according to the system of linear equations

$$\begin{aligned} &\dot{x}_i = V_i(\sin\theta_i^0 + z_i \cos\theta_i^0), \quad \dot{y}_i = V_i(\cos\theta_i^0 - z_i \sin\theta_i^0) \\ &\dot{z}_i = (V_i / R_i)\varphi_i, \quad |\varphi_i| \leqslant 1 \quad (z_i = \delta\theta_i) \end{aligned} \tag{1.4}$$

The game is considered in a time interval $[t_0, T]$, where T is not fixed and, in particular, can be infinite. The initial positions of P, E and E_1 are given by the vectors $\{x_i^0, y_i^0, z_i^0\}$ ($i = 1, 2, 3$). We will assume that

$$x_2^0 = x_3^0, \quad y_2^0 = y_3^0, \quad \theta_2^0 = \theta_3^0 \tag{1.5}$$

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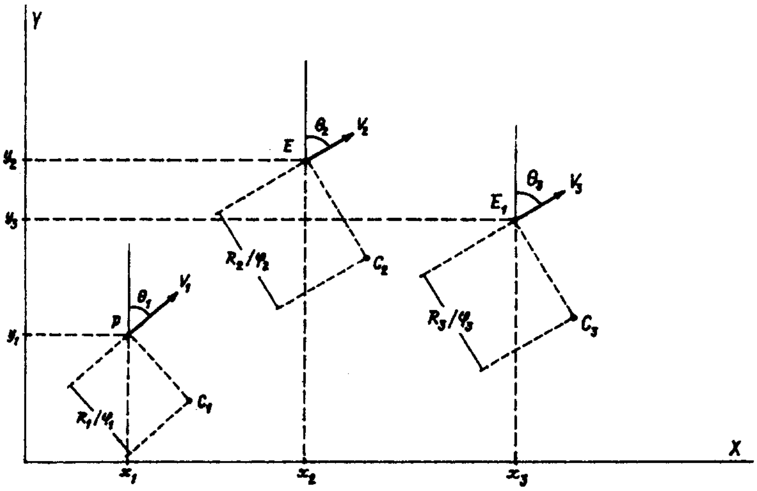

Fig. 1.

when $t = t_0$.

It is assumed that one of the evaders (it is unknown in advance which one) suddenly disappears from the game at an instant of time $t = t^*$ unknown in advance, where $t_0 \leqslant t^* < T$. Starting from this time a pursuit–evasion game between the pursuer and the remaining evader begins. The time t^* is a parameter of the problem under consideration. The solution of the problem depends explicitly on t^*, which is reflected in (7.1)–(7.3).

The basic payoff functional is given by

$$\gamma_1 = \min_{t^* \leqslant t < T} \{[x_i(t) - x_1(t)]^2 + [y_i(t) - y_1(t)]^2\}^{1/2} \tag{1.6}$$

The intermediate functional in the initial game has the form

$$\gamma_2 = \max\{|x_k(T_1) - x_1(T_1)|,\ |x_l(T_2) - x_1(T_2)|\} \tag{1.7}$$

$$y_k(T_1) = y_1(T_1), \quad y_l(T_2) = y_1(T_2)$$

where k and l can take only the following two pairs of values depending on the initial positions of E and E_1 and on θ_i^0: either $k = 2, l = 3$ or $k = 3, l = 2$.

In (1.6) i is equal to two or three depending on which of the evaders disappears from the game at $t = t^*$ (it is either equal to two if E remains in the game or to three if E_1 remains). In (1.7) the times $T = T_j$ ($j = 1, 2$) are defined according to [3] as the instants of contact between the extreme points $x_i(T_j)$ ($i = 2, j = 1$; $i = 3, j = 2$ or $i = 2, j = 2$; $i = 3, j = 1$) of the attainability domains of E and E_1 and the straight lines $y = y_1(T_1)$ and $y = y_1(T_2)$, respectively. It is assumed that the attainability domains of E and E_1 correspond to $T = T_j$ ($j = 1, 2$).

It is also assumed that the relations between V_i, R_i, θ_i^0 ($i = 1, 2, 3$) satisfy the covering conditions from [3] of the form

$$V_1 \geqslant V_2\left(\cos\theta_i^0 + \sqrt{2y_2^0|\sin\theta_i^0|/R_2}\right) \tag{1.8}$$

$$V_1^2(V_2\cos\theta_i^0 - V_1)/R_1 + V_2^2(V_1\cos\theta_i^0 - V_2)/R_2 \leqslant 0 \tag{1.9}$$

if $0 \leqslant \theta_i^0 \leqslant \pi/2$ or $3\pi/2 < \theta_i^0 \leqslant 2\pi$

$$V_1^2(V_2\cos\theta_i^0 - V_1)/R_1 + V_2^2(V_2 - V_1\cos\theta_i^0)/R_2 \leqslant 0 \tag{1.10}$$

if $\pi/2 \leqslant \theta_i^0 \leqslant 3\pi/2$.

The pursuer P aims to minimize the basic and auxiliary functionals γ_1 and γ_2, while the evaders try to maximize them.

2. FORMULATION OF THE PROBLEM

For the differential game (1.1)–(1.7) in which the parameters of P, E and E_1 satisfy the covering conditions (CC) (1.8)–(1.10), it is required to construct optimal positional strategies $U = U(t, X_*, Y_*, Z_*)$: $[t_0, T] \times R^3 \times R^3 \times R^3 \to \{\varphi_1\}$ for the pursuer P and $V_i = V_i(t, X_*, Y_*, Z_*)$: $[t_0, T] \times R^3 \times R^3 \times R^3 \to \{\varphi_1\}$ $(i = 2, 3)$ for the evaders E and E_1, which realize a saddle point of the game in the sense of the functional (1.6).

Here $X_* = \{x_1, y_1, z_1\}$, $Y_* = \{x_2, y_2, z_2\}$, $Z_* = \{x_3, y_3, z_3\}$, $\{\varphi_i\} = \{\varphi_i \in R^1\colon |\varphi_i| \leqslant 1\}$ $(i = 1, 2, 3)$.

Remark 1. The strategy of one of the evaders will be determined only in the interval $[t_0, t^*)$ according to the setting of the problem. However, as will be shown below, the choice of the strategy for each of the two evaders in $[t_0, t^*)$ will have a significant effect on the value of γ_1.

Remark 2. In (1.1)–(2.7) the condition for a saddle point to exist in a "little game" [2] is satisfied.

3. AUXILIARY CONSTRUCTIONS. CHANGE OF COORDINATES. ATTAINABILITY DOMAINS

To simplify the constructions below we will change the reference system as in [3], placing the origin at the initial position of P and directing the OY axis along the velocity vector of P and the OX axis to the right of $\mathbf{V}_1$. The positions of E and E_1 (which coincide at $t = t_0$) are given by a vector $(x_i^*, y_i^*, \theta_i^*)$ in the new reference system, where the components of the vector can be computed by means of formulae (4.1) in [3]. In what follows, for simplicity we shall omit the asterisk on the symbols for the actual positions of the objects and replace the symbols for the control functions φ_i $(i = 1, 2, 3)$ by u for P, and by v and v_1 for E and E_1, respectively.

In the new reference system the equations of motion (1.4) of P, E and E_1 take the form

$$\dot{x}_1 = V_1 z_1, \quad \dot{y}_1 = V_1, \quad \dot{z}_1 = (V_1 / R_1) u, \quad |u| \leqslant 1 \tag{3.1}$$

$$\dot{x}_i = V_2 (\sin\theta_i^0 + z_i \cos\theta_i^0), \quad \dot{y}_i = V_2 (\cos\theta_i^0 - z_i \sin\theta_i^0) \tag{3.2}$$

$$\dot{z}_i = (V_2 / R_2) v_i, \quad |v_i| \leqslant 1$$

Expressions (1.6) and (1.7) for the payoff functional remain unchanged. It was shown in [3, 4] that the attainability domains $G^{(i)}(t, T)$ of P, E and E_1 will be rectilinear intervals $r_i(t, T)$ perpendicular to $\mathbf{V}_i$ $(i = 1, 2, 3)$ (Fig. 2) with

$$r_i(t, T) = (V_i (T - t))^2 / (2R_i)$$

Typical positions of P, E and E_1 at the initial time $t = t_0$ and their attainability domains at a time $T > t_0$ are shown in Fig. 2.

Assuming the control functions of P, E and E_1 to be constant in the time interval $[t_0, T]$, we can integrate (3.1) and (3.2) and find the coordinates (x_i, y_i, z_i) $(i = 1, 2, 3)$ of these objects when $t = T$ given that they are at positions $\{t_0, x_i^0, y_i^0, z_i^0\}$ respectively, at $t = t_0$. The corresponding values are given in [3] (formulae (5.3) and (5.4)).

4. CONDITIONS FOR COVERING THE ATTAINABILITY DOMAINS OF E AND E_1 BY THE ATTAINABILITY DOMAIN OF P

According to (1.5), when $t = t_0$ the positions of the two evaders E and E_1 coincide at the point $\{t_0, x_2^0, y_2^0, z_2^0\} = \{t_0, x_3^0, y_3^0, z_3^0\}$. Suppose that P is at $\{t_1, x_1, y_1, z_1\}$ at time $t = t_1$, where $t_0 < t_1 \leqslant t^*$, while E and E_1 are at $\{t_1, x_2, y_2, z_2\}$ and $\{t_1, x_3, y_3, z_3\}$, respectively.

For these positions we consider the attainability domains of P, E and E_1 at times $T_1 > t_1$ and $T_2 > t_1$ corresponding to an arbitrary value of θ_i^0, where $0 < \theta_i^0 < \pi/2$.

We denote the extreme points of the attainability domains of P, E and E_1 corresponding to the control interactions $u = \pm 1$, $v = \pm 1$, $v_1 = \pm 1$ and times $T = T_j$ $(j = 1, 2)$ by $A_1(T_j)$, $B_1(T_j)$; $A_2(T_j)$, $B_2(T_j)$ and $A_3(T_j)$, $B_3(T_j)$ respectively (Fig. 3).

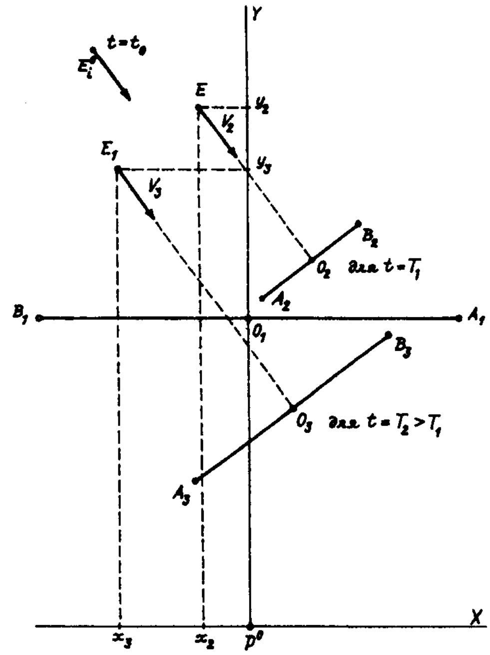

Fig. 2.

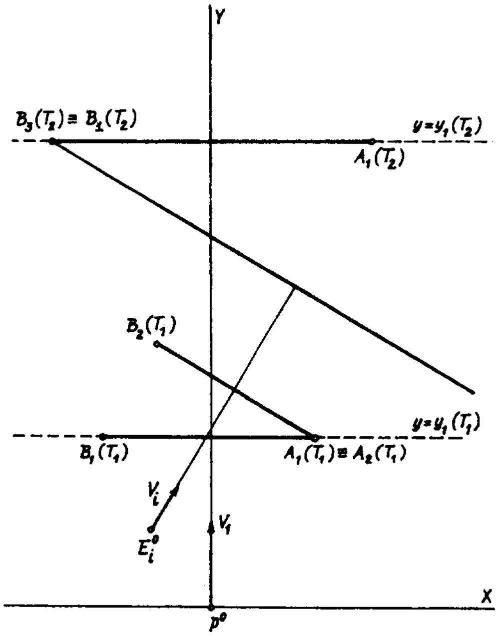

Fig. 3.

Suppose that for $T = T_1$ the geometric coordinates of one of the extreme points of the attainability domain of E_1, for example, $A_2(T_1)$ $\{x_2(T_1), y_2(T_1)\}$ lie on the straight line $y = y_1(T_1)$ and the inequality

$$x_2(T_1) - x_1(T_1) \leqslant 0 \tag{4.1}$$

holds, where $x_2(T_1)$ is the abscissa of the point $A_2(T_1)$ in the attainability domain of E_1 and $x_1(T_1)$ is the abscissa of the point $A_1(T_1)$ in the attainability domain of P closest to $A_2(T_1)$.

Next, suppose that an extreme point $B_3(T_2) = \{x_3(T_2), y_3(T_2)\}$ of the attainability domain of E belongs to the straight line $y = y_1(T_2)$ at $T = T_2$ (Fig. 3) and the inequality

$$x_3(T_2) - x_1(T_2) \geqslant 0 \tag{4.2}$$

holds, where $x_3(T_2)$ is the abscissa of the point $B_3(T_2)$ in the attainability domain of E and $B_1(T_2)$ is the abscissa of the point $B_3(T_2)$ in the attainability domain of P closest to $B_3(T_2)$.

We shall assume that at least one of inequalities (4.1) and (4.2) is strict.

Such a configuration of the attainability domains of P, E and E_1 at $T = T_1$ and $T = T_2$ whose positions at $t = t_1$ are given above will be referred to as a situation of covering the attainability domain of P by those of E and E_1 in a "one against two" game. The conditions relating the parameters of P, E and E_1 for which the attainability domains realize a "situation of covering" will be called the covering conditions (*CC*) in a "one against two" game. These conditions are similar to the *CC* in [3] and have the form (1.8)–(1.10).

5. AUXILIARY PROBLEM 1

We will first consider the problem of constructing an optimal positional strategy (OPS) of P, E and E_1 on $[t_0, t^*)$ whose payoff functional is given by (1.6).

Formulation of the Auxiliary Problem 1. For the differential game (1.1)–(1.6) in which the parameters and the initial positions of P, E and E_1 satisfy the *CC* in a "one against two" game we need to construct the OPS

$$U^1 = U^1(t, X_*, Y_*, Z_*):\ [t_0, t^*) \times R^3 \times R^3 \times R^3 \to \{\varphi_1\}$$

of the pursuer P and the OPSs

$$V_i^1 = V_i^1(t, X_*, Y_*, Z_*):\ [t_0, t^*)\times R^3\times R^3\times R^3 \to \{\varphi_1\} \quad (i=2,3)$$

of the evaders E and E_1 in a time interval $[t_0, t^*)$, where $t = t^*$ is unknown in advance, which realize a saddle point in the game (1.1)–(1.5), (1.7).

From the definition of a situation of covering for P, E and E_1 it follows that for some position $\{t, X_*, Y_*, Z_*\}$ of the game (1.1)–(1.6) for which the CC are satisfied, the following inequalities hold, relating the ordinates of the points of the attainability domains

$$y_1(T_1) = y_k(T_1), \quad y_1(T_2) = y_l(T_2) \tag{5.1}$$

where either $k = 2, l = 3$ or $k = 3, l = 2$, so T_j $(j = 1, 2)$ is satisfied according to [3].

It can be shown that only the x coordinates $x_i(T_j)$ $(i = 1, 2, 3; j = 1, 2)$ of the extreme points of the attainability domains of P, E and E_1 at the times $T = T_j$ $(j = 1, 2)$ when these points lie on the straight lines $y = y_1(T_j)$ $(j = 1, 2)$ are important for constructing the strategies U^1 and V_i^1 $(i = 2, 3)$.

We introduce the notation

$$s_i^j = s_i(t, T_j) = x_i(T_j) - x_1(T_j), \quad i = 2,3; \quad j = 1,2 \tag{5.2}$$

Consider strategies $U^1 = U^1(t, X_*, Y_*, Z_*)$ and $V_i^1 = V_i^1(t, X_*, Y_*, Z_*) \equiv V_i^1(t)$ of the form

$$U^1(t, X_*, Y_*, Z_*) = \begin{cases} \operatorname{sign}(s_k^1 + s_l^2), & \text{if } |s_k^1| \neq |s_l^2| \\ [-1, +1], & \text{if } |s_k^1| = |s_l^2| \end{cases} \tag{5.3}$$

$$V_i^1 \equiv V_i^1(t) = (-1)^i, \quad i = 2,3, \quad t \in [t_0, t^*) \tag{5.4}$$

where the subscripts k and l can only take the following two pairs of values depending on the initial position of E and E_1 and the angle θ_i^0: either $k = 2, l = 3$ or $k = 3, l = 2$.

Now suppose that P, E and E_1 in the auxiliary game (1.1)–(1.5), (1.7) move subject to the strategies (5.3) and (5.4). The positions of the game under consideration for $t_0 \leqslant t \leqslant t^*$ such that

$$s_k^1 = -s_l^2 \tag{5.5}$$

form a focal surface S_1 similar to that considered in [10, 11]. The functional γ_2 may increase as P moves on S_1. However, it can be shown that the strategy $U^1(t, X_*, Y_*, Z_*)$ defined by (5.3) guarantees that this growth will be the slowest possible. Those positions of the game (1.1)–(1.5), (1.7) for which (5.5) is not satisfied belong to the regular domain of the game under consideration. It can be shown that the programming strategies given by (5.4) are the desired optimal strategies of E and E_1 in $[t_0, t^*)$. This follows from the fact that the strategies (5.4) for any admissible strategies of P lead to the quickest violation of the situation of covering in a "one against two" game and provide the maximum of γ_2 with respect to V_i $(i = 2, 3)$.

Taking the above discussion into account and analysing how the attainability domains of P, E and E_1 change with time under the strategies U^1 and V_i^1 $(i = 2, 3)$, it can be shown that strategies (5.3) and (5.4) provide a solution of Auxiliary Problem 1 and satisfy the equality

$$\max_{V_2^1, V_3^1} \min_{U^1} \gamma_2 = \min_{U^1} \max_{V_2^1, V_3^1} \gamma_2 = \gamma_2^* \tag{5.6}$$

6. AUXILIARY PROBLEM 2

At $t = t^*$ the game (1.1)–(1.5), (1.7) turns into a pursuit–evasion game (1.1)–(1.6) between the pursuer and the evader remaining in the game (E or E_1).

We introduce the notation

$$\varepsilon_1(t, x) = s_k(t, T_1), \quad \varepsilon_2(t, x) = s_l(t, T_2) \tag{6.1}$$

where $x = \{X_*, Y_*, Z_*\}$ and, as in (5.3), the subscripts k and l can take only two pairs of values depending on the initial positions of E and E_1 and on $k = 2, l = 3$: either $\theta^0{}_i$ or $k = 3, l = 2$.

Let $x^* = \{X^*_*, Y^*_*, Z^*_*\}$ be the coordinates of P, E and E_1 at the time $t = t^*$ of termination of the auxiliary game 1 and let $T = T_j$ ($j = 1, 2$) in (6.1) be computed for the position $\{t^*, x^*\}$ in accordance with [3]. It can be shown that the strategies (5.3) and (5.4) in the game (1.1)–(1.6) ensure that

$$\varepsilon_1(t^*, x^*) = \varepsilon_2(t^*, x^*) = \gamma_2^* \tag{6.2}$$

Formulation of Auxiliary Problem 2. For the differential game (1.1)–(1.6) in which the initial positions of P and E (or E_1) coincide with the final positions of P, E and E_1 at $t = t^*$ we need to construct in $[t^*, T)$ an OPS

$$U^2 = U^2(t, X_*, x_i, y_i, z_i)\colon\ [t^*, T) \times R^3 \times R^3 \to \{\varphi_1\}$$

of the pursuer P and a positional strategy

$$V_i^2 = V_i^2(t, X_*, x_i, y_i, z_i)\colon\ [t^*, T) \times R^3 \times R^3 \to \{\varphi_i\}$$

of the evader remaining in the game ($i = 2$ if E remains in the game and $i = 3$ if E_1 remains) providing a saddle point in the game (1.1)–(1.6).

For $t = t^*$ the game (1.1)–(1.5), (1.7) turns into problem (1.1)–(1.6) with one pursuer P and one evader E or E_1, considered in [3]. In this game the strategy of the remaining evader (E or E_1) must be a positional strategy, because such a strategy guarantees the best result for the evader and enables it to "punish" the pursuer for any deviations from its optimal strategy.

As in the construction of the OPS (5.3) and (5.4) for P, E and E_1 in Auxiliary Problem 1 it can be shown that the desired positional strategies of P and E (E_1) in Auxiliary Problem 2 have the form

$$U^2(t, X_*, x_i, y_i, z_i) = \begin{cases} \operatorname{sign}(s_i^1 + s_i^2), & \text{if } |s_i^1| \neq |s_i^2| \\ [-1, +1], & \text{if } |s_i^1| = |s_i^2| \end{cases} \tag{6.3}$$

$$V_i^2(t, X_*, x_i, y_i, z_i) = \begin{cases} (-1)^q \operatorname{sign}(s_i^1 + s_i^2), & \text{if } |s_i^1| \neq |s_i^2| \\ [-1, +1], & \text{if } |s_i^1| = |s_i^2| \end{cases} \tag{6.4}$$

In (6.3) and (6.4) $i = 2$ if E remains in the game and $i = 3$ if E_1 remains. In (6.4) $q = 2$ if $0 \leqslant \theta^0_i < \pi/2$ or $3\pi/2 < \theta^0_i \leqslant 2\pi$ and $q = 1$ if $\pi/2 < \theta^0_i \leqslant 3\pi/2$.

We assume that P, E (E_1) move, respectively, according to the strategies U^2 and V_i^2 in the interval $[t^*, T)$. The positions of the game (1.1)–(1.6) for which the equality

$$|s_i^2| = |s_i^3| \tag{6.5}$$

holds for $t^* \leqslant t < T$ from a singular set S_2, which is a dispersive surface.

Auxiliary Problem 2 was solved in [3]. It can be shown that

$$\max_{V_i^2} \min_{U^2} \gamma_1 = \min_{U^2} \max_{V_i^2} \gamma_1 = \gamma_1^* \tag{6.6}$$

holds for Auxiliary Game 2. From the solution of this problem it follows that U^2 and V_i^2 guarantee that the result γ_1^* of the game will be

$$\gamma_1^* = \gamma_2^* = \gamma^* \tag{6.7}$$

7. OPTIMAL STRATEGIES OF P AND E (E_1) IN THE ORIGINAL GAME

The positional strategies $U(t, X_*, Y_*, Z_*)$ and $V_i(t, X_*, Y_*, Z_*)$ ($i = 2$ or 3) representing the solution of the original game (1.1)–(1.7) can be constructed on the basis of the solution of Auxiliary Problems 1 and 2 and have the form

$$U(t, X_*, Y_*, Z_*) = \begin{cases} U^1(t, X_*, Y_*, Z_*) & \text{for} \quad t_0 \le t < t^* \\ U^2(t, X_*, x_i, y_i, z_i) & \text{for} \quad t^* \le t < T \end{cases} \tag{7.1}$$

$$V_i(t, X_*, Y_*, Z_*) = \begin{cases} V_i^1(t) & \text{for} \quad t_0 \le t < t^*, \quad i = 2,3 \\ V_i^2(t, X_*, x_i, y_i, z_i) & \text{for} \quad t^* \le t < T, \quad i = 2 \text{ or } 3 \end{cases} \tag{7.2}$$

where U^m and $V^m{}_i$ $(m = 1, 2; i = 2, 3)$ are given by (5.3), (6.3) and (5.4), (6.4), respectively.

Suppose that for some position $\{t, X_*, Y_*, Z_*\} = \{t, x\}$ of the game (1.1)–(1.7) the CC are satisfied in the "one against two" game. We assume that P, E and E_1 move according to strategies (7.1) and (7.2). Suppose that P, E and E_1 are at positions $\{t^*, X^*_*, Y^*_*, Z^*_*\} = \{t^*, x^*\}$ at the time $t = t^*$ $(t_0 \le t^* < T)$ (unknown in advance) when one of the evaders (E or E_1) disappears from the game. Then it follows from (6.2) and (6.7) that the optimal solution $\varepsilon(t, x)$ of the original game (1.1)–(1.7) has the form

$$\varepsilon(t, x) = \varepsilon(t^*, x^*) = \gamma^* \tag{7.3}$$

This solution is guaranteed by the two strategies (7.1) and (7.2).

Remark 3. As in [3], we have the identity $T_2 \equiv T_2$ whenever θ_i^0 or $\theta_i^0 = \pi$. This means that problem (1.1)–(1.7) turns into a game with fixed final time $T = T_1 = T_2$ and terminal functional. In these cases, as above and in [10, 11], singular scalar manifolds of type S_1 (a focal surface) and type S_2 (a dispersive surface) will appear in $[t_0, t^*)$. It can be shown that in these cases strategies (7.1) and (7.2) will also be optimal and will guarantee the result of the game (1.1)–(1.7) to be equal to (7.3).

8. CONSTRUCTION OF THE SET OF POSITIONS OF THE OBJECT E (E_1) FROM WHICH IT CANNOT ESCAPE THE PURSUER P FOR FIXED VALUES OF THE PARAMETERS OF BOTH OBJECTS

Let the values of the parameters V_1, R_1 of P and V_i, R_i, θ_i^0 $(i = 2, 3)$ of E and E_1 be given. We recall that according to (1.2) and (1.5) these values of the parameters are equal for E and E_1. We assume that a situation of covering is realized in the game, and so the CC are satisfied for the given parameter values. We fix $t = T_1$ arbitrarily.

It is required to determine the positions (x_i^0, y_i^0) $(i = 1, 2)$ of E and E_1 at $t_0 = 0$ belonging to the set of positions K from which E (or E_1) cannot avoid being caught by P in the game (1.1)–(1.7).

We put

$$y^* = R_i[(V_1 - V_i \cos\theta_i^0) / V_i]^2 / (2|\sin\theta_i^0|) \tag{8.1}$$

It follows from (8.1) that the initial coordinate y_i^0 of E (E_1) satisfies the inequality

$$y_i^0 \le y^* \tag{8.2}$$

As has been shown above, up to $t = T_1$ the attainability domains of P, E and E_1 turn out to be straight line intervals A_iB_i orthogonal to the vectors $\mathbf{n}_i = \mathbf{V}_iT_1$ $(i = 1, 2, 3)$, respectively. In accordance with the situation of covering at $t = T_1$ the coordinates of the extreme points of the attainability domains of P and E (or E_1) coincide

$$x_1(T_1) = x_i(T_1), \qquad y_1(T_1) = y_i(T_1) \tag{8.3}$$

where $i = 2$ or 3 depending on the given value of θ_i^0.

Using the CC, we can determine (x_i^0, y_i^0) at $t_0 = 0$ for any y_i^0 satisfying (8.2), as described in [3].

By varying the initial value y_i^0 between 0 and y^* one can construct a curve K (Fig. 4) representing the set of initial positions of E (E_1) from which E (or E_1) cannot escape the pursuer P in the game (1.1)–(1.7).

We introduce the notation

$$a = (V_i^2 \cos\theta_i^0) / (2R_i) - V_i^2 / (2R_1), \qquad b = V_i \sin\theta_i^0$$

It can be shown that the desired curve K is described by the following equations depending on the given value of θ^0_i

$$x(y) = -aT_1^2 - bT_1 \quad (\text{or } x(y) = aT_2^2 - bT_2) \tag{8.4}$$

where $0 \leqslant y \leqslant y^*$ and T_j ($j = 1, 2$) can be computed according to [3] separately for each y.

9. CONSTRUCTION OF THE DOMAIN OF INITIAL POSITIONS OF E (E_1) FROM WHICH E (OR E_1) CANNOT ESCAPE THE PURSUER P

Let the following parameters of E and E_1 be given: the velocities V_i, the radius of curvature R_i of the trajectory and the angle θ^0_i between the velocity vector $\mathbf{V}_i$ and the OY axis ($i = 2, 3$), for example, $0 \leqslant \theta^0_i < \pi/2$. We recall that according to (1.2)–(1.5) the parameters and initial positions of E and E_1 as well as the values θ^0_i ($i = 1, 2$) coincide.

For any initial coordinate $y^0_i \geqslant 0$ of E and E_1, using (1.8) from the CC, we specify the velocity V_1. Depending on the angle θ^0_i from (1.9) or (1.10) we can determine the maximum possible radius of curvature R_1 of the trajectory of P, which will be denoted by $R_1^{\max}$. We introduce the notation

$$a^* = (V_i^2 \cos\theta_i^0)/(2R_i) - V_1^2/(2R_1^{\max}), \quad b^* = V_i \sin\theta_i^0$$

where i is equal to two or three depending on which of the objects E or E_1 remains in the game for $t > t^*$.

From the procedure for constructing the curve K in Section 8 we can draw the following conclusion. If the initial position of E and E_1 in the "one against two" game belongs to the domain bounded by the lines $y = y^*$ and $y = 0$ lying to the left of the curve $x(y) = -a^*T_1^2 - b^*T_1$ (or $x(y) = a^*T_2^2 - b^*T_2$), where $0 \leqslant y \leqslant y^*$, then the pursuit–evasion game of "one against one" results in the capture of E (or E_1). Otherwise E (or E_1) avoids being captured in the "one against one" game. This domain is the desired domain of initial positions of E and E_1, in which E (or E_1) cannot avoid being captured by P. In Fig. 4 such a domain is presented for $V_i = 2$, $R_i = 3$, $V_1 = 6$ and $\theta^0_i = \pi/6$.

It can be shown that the domain consists of a bundle of curves K constructed for various values of R_1 from the interval $0 \leqslant R_1 \leqslant R_1^{\max}$. Curve K corresponding to $R_1 = R_1^{\max}$ is the left boundary of the desired domain of initial positions of E and E_1 (Fig. 4).

Similar domains of initial positions of E and E_1 can be constructed for other values of θ^0_i.

Remark 4. On the basis of the results of this paper one can solve the problem of constructing a positional strategy or the evader remaining in the "one against two" pursuit–evasion game which guarantees that the evader will stay away from the pursuer by a given distance and arrive in a prescribed set in which the game terminates.

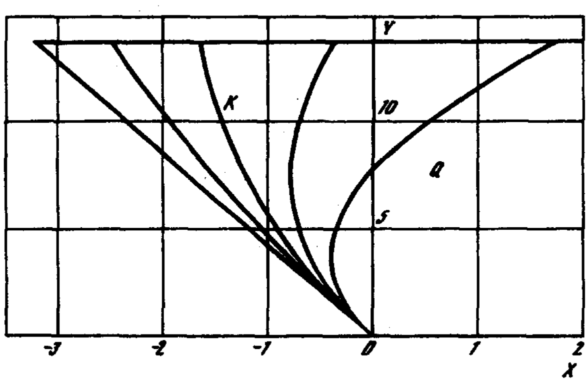

Fig. 4.

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